Last time

- finitely generated $\underbrace{\text{torsion-free}}$ abelian groups are

$$G \cong \mathbb{Z}^r$$

$ng = 0$ for $n > 0 \Rightarrow g = 0$

- finitely generated abelian groups are

$$G \cong \mathbb{Z}^r \times \mathrm{Tors}(G)$$

$\{g \in G \mid ng = 0 \text{ for some } n > 0\}$

- today, we will show that any finitely generated abelian group is

$$\cong \mathbb{Z}^r \times \underbrace{\mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p_k^{d_k}\mathbb{Z}}_{\text{finite group}}$$

- key lemma from last time: any s.e.s.

$$0 \to K \to G \to \mathbb{Z}^l \to 0 \quad \text{splits}$$

Today: understanding torsion abelian groups

- G_1, G_2 abelian groups $\rightsquigarrow G_1 \times G_2$ component-wise addition

- $G_1, G_2, \ldots\ldots, G_n, \ldots$ abelian groups

direct product $\prod_{k=1}^{\infty} G_k = \{ (g_1, g_2, \ldots, g_n, \ldots) \mid g_i \in G_i \}$ component-wise addition

$\cup$

direct sum $\bigoplus_{k=1}^{\infty} G_k = \{ (g_1, g_2, \ldots, g_n, \ldots) \mid g_i \in G_i \text{ s.t. } g_i = 0 \ \forall i \text{ large enough} \}$

if $\exists N$ s.t. $G_{N+1} = G_{N+2} = \ldots = \{0\}$, then $\prod = \bigoplus = G_1 \times \ldots \times G_N$

Example $G_1 = G_2 = \ldots\ldots = \mathbb{Z}/2\mathbb{Z}$

$$\prod_{k=1}^{\infty} \mathbb{Z}/2\mathbb{Z} = \{ (\underset{g_0}{0}, \underset{g_1}{0}, \underset{g_2}{1}, \underset{g_3}{0}, \underset{g_4}{0}, 1, 1, 0, \ldots, 1, 0, \ldots) \} \xleftrightarrow{\sim} [0, 2] \qquad \sum_{i=0}^{\infty} g_i \cdot \frac{1}{2^i}$$

$$\bigoplus_{k=1}^{\infty} \mathbb{Z}/2\mathbb{Z} = \{ (\underset{g_0}{0}, \underset{g_1}{0}, \underset{g_2}{1}, \underset{g_3}{0}, \underset{g_4}{0}, 1, 1, 0, 0, 0, 0, \ldots, 0, \ldots) \} \xleftrightarrow{\sim} \mathbb{N} \qquad \sum_{i=0}^{\infty} g_i \cdot 2^i$$

<u>Def</u>: G be a abelian group ; p prime number

Last time: $\mathrm{Tors}_n(G) \subseteq G$ (n-th torsion subgroup)

Today: $A_p(G) = \bigcup_{k=1}^{\infty} \mathrm{Tors}_{p^k}(G)$ (p-torsion subgroup)

$= \{ g \in G \mid \text{order of } g \text{ is a power of } p \}$

is a subgroup of G
proof like last time

$$\left(\begin{array}{l} \mathrm{ord}(g) = p^k \\ \mathrm{ord}(h) = p^L \end{array} \Rightarrow \begin{array}{l} p^{\max(k,L)} \cdot g = 0 \\ p^{\max(k,L)} \cdot h = 0 \end{array} \Rightarrow p^{\max(k,L)}(g+h) = 0 \right)$$

Lemma: G an abelian group

$$\mathrm{Tors}(G) \cong \bigoplus_{\text{primes } p} A_p(G)$$

$$g_2+g_3+g_5+\dots \xleftarrow{\ \Phi\ } (g_2, g_3, g_5, 0, \dots)$$

$$(g_2, g_3, g_5, 0, \dots) = (g_2, 0, 0, 0, \dots) + (0, g_3, 0, 0, \dots) + (0, 0, g_5, 0, \dots) + \dots$$

well-defined only because $(g_2, g_3, g_5, \dots)$ because all g_p's are 0 from a point onwards

Proof: injectivity: suppose $\Phi\big((0,\dots,0, g_{p_1}, 0, g_{p_2}, 0, \dots, 0, g_{p_k}, 0, \dots)\big) = 0$

for $(p_1, \dots, p_k)$ distinct primes,

$$\Downarrow$$

$$g_{p_1} + g_{p_2} + \dots + g_{p_k} = 0 \quad \Big| \cdot p_2^{d_2} \cdots p_k^{d_k}$$

where

$$p_1^{d_1} g_{p_1} = 0$$
$$\vdots$$
$$p_k^{d_k} g_{p_k} = 0$$

$$\Downarrow$$

$$p_2^{d_2} \cdots p_k^{d_k}\, g_{p_1} = 0 \qquad (b*)$$

$$\text{but } p_1^{d_1}\, g_{p_1} = 0 \qquad (a*)$$

$$\exists\ a \cdot p_1^{d_1} + b\, p_2^{d_2} \cdots p_k^{d_k} = 1$$

$$(\Downarrow)$$

$$1 \cdot g_{p_1} = 0$$

Similarly, $g_{p_2}, \dots, g_{p_k} = 0 \implies \Phi$ is injective

surjectivity: $g \in \mathrm{Tors}(G) \implies \exists\, n \in \mathbb{N}$ s.t. $ng = 0$

$n = p_1^{d_1} \cdots p_k^{d_k}$, $p_1, \dots, p_k$ distinct primes

$n_1 = \frac{n}{p_1^{d_1}}$

$$\left.\begin{array}{l} n_1 = \frac{n}{p_1^{d_1}} \\ \vdots \\ n_k = \frac{n}{p_k^{d_k}} \end{array}\right\} \text{coprime} \Longrightarrow \begin{array}{l} \exists\, a_1, \dots, a_k \in \mathbb{Z} \\ \text{s.t. } a_1 n_1 + \dots + a_k n_k = 1 \end{array}$$

$$g_1 = n_1 g \rightsquigarrow p_1^{d_1} g_1 = p_1^{d_1} n_1 g = ng = 0 \Rightarrow g_1 \in A_{p_1}(G)$$

$$\vdots$$

$$g_k = n_k g \rightsquigarrow p_k^{d_k} g = p_k^{d_k} n_k g = ng = 0 \Rightarrow g_k \in A_{p_k}(G)$$

$$a_1 g_1 + \dots + a_k g_k = (a_1 n_1 + \dots + a_k n_k) g = 1 \cdot g = g$$

$$\Downarrow$$

$$g = \Phi((0, \dots, 0, \underset{A_{p_1}(G)}{\underset{\cap}{g_1}}, 0, \dots, 0, \underset{A_{p_k}(G)}{\underset{\cap}{g_k}}, 0, \dots))$$

$\square$

Def: a group is called a p-group if all of its elements have order p^k, for arbitrary k

Ex: $A_p(G)$ is a p-group, $\forall$ abelian group G

$\mathbb{Z}/p^{k_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{k_d}\mathbb{Z}$ is also a p-group

$\mathbb{Z}/p\mathbb{Z} \times \dots \times \mathbb{Z}/p\mathbb{Z} \times \dots = \prod_{k=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a p-group.

Prop: a finite abelian group G is a p-group

$$\Updownarrow$$

$$|G| = p^k \quad \text{for some } k$$

Proof: $\Uparrow$ Lagrange's theorem

$\Downarrow$ immediately from Lemma below

(if $\exists$ $q \neq p$ which divides $|G|$ $\Rightarrow$ $\exists$ an element order of order q)

Lemma: if a prime p divides $|G|$, finite abelian group
then G has an element of order p.

Proof: induction on $|G|$; base case is trivial, let's do induction step

take $h \in G$, $d = \text{ord}(h)$

- $p \mid d$, $\frac{d}{p} \cdot h$ has order p and we are done
- $p \nmid d$ $\longrightarrow$ G is generated by H

$$\overset{!!}{G \simeq \mathbb{Z}/d\mathbb{Z}}$$
$$h \leftrightarrow 1$$

($|G| = d$ is not divisible by p, contradiction)

$\longrightarrow$ H = subgroup of G generated by h;

$$H \cong \mathbb{Z}/d\mathbb{Z}$$

$$H \subsetneq G \rightsquigarrow |G/H| = \frac{|G|}{d}$$ is a multiple of p

p divides

p does not divide

Induction hypothesis $\Longrightarrow$ G/H has an element $[g]$ of order p

Induction hypothesis $\Rightarrow \quad /_H$

$$pg \in \underset{\mathbb{Z}/d\mathbb{Z}}{\overset{}{H}} \Rightarrow dpg = 0 \Rightarrow \operatorname{ord}(g) \mid dp$$

$$H \cong \mathbb{Z}/d\mathbb{Z}$$

- $p \mid \operatorname{ord}(g) \Rightarrow \frac{\operatorname{ord}(g)}{p} \cdot g$ will have order p ✓
- $p \nmid \operatorname{ord}(g) \Rightarrow \operatorname{ord}(g) \mid d \Rightarrow$

$$\begin{array}{l} dg = 0 \in H \\ pg = 0 \in H \\ \hline 1 \cdot g = 0 \in H \end{array} \quad (+)$$

contradiction, because $[g]$ has order p in $G/_H$, not order 1 $\square$

Thm: any finite abelian p-group G is

$$G \cong \mathbb{Z}/p^{d_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{d_k}\mathbb{Z}$$

for some $d_1, \dots, d_k \in \mathbb{N}$ (which are unique up to permutation).

$\Downarrow$

Cor: any finite abelian group G is

$$G \underset{\text{first half of class}}{\overset{\text{Lemma from}}{\cong}} A_{p_1}(G) \times \dots \times A_{p_k}(G) \overset{\text{Thm above}}{\cong} \mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \mathbb{Z}/p_2^{e_2}\mathbb{Z} \times \dots \times \mathbb{Z}/p_k^{f_k}\mathbb{Z}$$

for $p_1, \dots, p_k$ distinct primes

$$\begin{array}{ccccccc} \times & & \times & & & & \times \\ \vdots & & \vdots & & & & \vdots \\ \times & & \times & & & & \times \\ \mathbb{Z}/p_1^{d_\alpha}\mathbb{Z} & \times & \mathbb{Z}/p_2^{e_\beta}\mathbb{Z} & \times & \dots & \times & \mathbb{Z}/p_k^{f_\gamma}\mathbb{Z} \end{array}$$

$G = \operatorname{Tors}(G)$ because G is finite product is finite because $A_p(G)=\{0\}$ for p large enough

<u>Proof</u> of Thm: by induction on $|G|$

let p^{d_1} be the maximal order of an element of G

pick $h \in G$ of order p^{d_1}

$$\mathbb{Z}/p^{d_1}\mathbb{Z} \cong H = \text{subgroup generated by } h \subseteq G$$

$$0 \to H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

order divides power of p, by Lemma above

$H \cong \mathbb{Z}/p^{d_1}\mathbb{Z}$

$G/H \cong \mathbb{Z}/p^{d_2}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{d_k}\mathbb{Z}$ by induction hypothesis

$$\boxed{d_1 \geq d_2, \dots, d_k} \text{ because } \operatorname{ord}_{G/H}([g]) \leq \operatorname{ord}_G(g) \leq p^{d_1}$$

$1 \rightsquigarrow h$

$$0 \to \mathbb{Z}/p^{d_1}\mathbb{Z} \xrightarrow{f} G \xrightarrow{g} \mathbb{Z}/p^{d_2}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{d_k}\mathbb{Z} \to 0$$

ψ

It suffices to show that s.e.s. above will split

pick $x_2, \dots, x_k \in G$, s.t. $x_i \in g^{-1}((0,0,\dots,0,1,0,\dots,0))$

i-th position

$\Downarrow$

$$p^{d_i} x_i \in \operatorname{Ker} g = \operatorname{Im} f$$

$$\Downarrow$$

$$p^{d_i}x_i = p^{s_i}t_i h \quad \text{where } p \nmid t_i,\ s_i \in \mathbb{N}$$

$$\Downarrow$$

but $d_1 \geq d_2, \ldots, d_k$

$$0 = p^{d_1}x_i = p^{d_1-d_i+s_i}t_i h$$

$$\Downarrow$$

$$p^{d_1} = \text{order}(h) = \text{order of } (t_i h) \mid p^{d_1-d_i+s_i}$$

prove it yourself (it's because G is a p-group)

$$\Downarrow$$

$$d_1 \leq d_1 - d_i + s_i$$

$$\Downarrow$$

$$d_i \leq s_i$$

$$\text{So } p^{d_i}x_i = p^{s_i}t_i h = p^{d_i}\left(p^{s_i-d_i}t_i h\right)$$

Define $x_i' = x_i - p^{s_i-d_i}t_i h \implies p^{d_i}x_i' = 0$

$$\begin{tikzcd} G \arrow[r, "g"] & \mathbb{Z}/p^{d_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{d_k}\mathbb{Z} \arrow[l, bend left, "\psi"] \end{tikzcd}$$

$$\psi\left((\alpha_2, \ldots, \alpha_k)\right) = \alpha_2 x_2' + \cdots + \alpha_k x_k'$$

is well-defined because $p^{d_i} = 0$ in $\mathbb{Z}/p^{d_i}\mathbb{Z}$ and $p^{d_i}x_i' = 0$ in G

$$g \circ \psi\left((\alpha_2, \ldots, \alpha_k)\right) = g\left(\alpha_2 x_2' + \cdots + \alpha_k x_k'\right)$$
$$= \alpha_2 g(x_2') + \cdots + \alpha_k g(x_k')$$

$$= \alpha_2(1,0,\dots,0) + \dots + \alpha_k(0,\dots,0,1)$$
$$= (\alpha_2,\dots,\alpha_k)$$

$\square$

- G a finitely generated abelian group

$$\Downarrow$$

$$G \cong \mathbb{Z}^r \times \mathrm{Tors}(G)$$

why?. See below

- since $\mathrm{Tors}(G)$ is a finite torsion abelian group,

$$\mathrm{Tors}(G) \cong A_{p_1}(G) \times \dots \times A_{p_k}(G) \cong \mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p_k^{d_k}\mathbb{Z}$$

G is fin. gen abelian $\Longrightarrow$ $\mathrm{Tors}(G)$ is finite

step 1 $\searrow$ $\nearrow$ step 2

does not hold for non-abelian

any subgroup of G is fin. gen

Step 1: $H \subseteq G$ and G is fin. gen. abelian, then H is fin. gen

Step 2: $\mathrm{Tors}(G)$ is fin. gen. abelian

take generators $g_1,\dots,g_k$ of $\mathrm{Tors}(G)$

$$\Downarrow$$

$d_1 g_1 = \dots = d_k g_k = 0$

$$\mathrm{Tors}(G) = \{ a_1 g_1 + \dots + a_k g_k \mid a_1 \in \{0, \dots, d_1 - 1\}, \dots, a_k \in \{0, \dots, d_k - 1\} \}$$

$\Downarrow$

Tors(G) is finite

Prop: $H \subseteq G$ and G is fin. gen. abelian, then H is fin. gen

<u>Proof</u>: induction on $n :=$ the minimal # of generators of G

base case $n = 1 \rightsquigarrow G \cong \mathbb{Z}$ or $\mathbb{Z}/k\mathbb{Z}$ (Exercise)

induction step: G is generated by $g_1, \dots, g_n$

$\cup\!\!\!\!\!\cdot$ (sub)

G' is generated by $g_1, \dots, g_{n-1}$

$$0 \to G' \xrightarrow{\alpha} G \xrightarrow{\beta} G/G' \to 0$$

G/G' has only one generator, namely $[g_n]$

$$\begin{array}{c} G \\ \cup \\ H \end{array}$$

define $K = H \cap G' \subseteq G'$ and define $L = \beta(H) \subseteq G/G'$

as last time, induces a s.e.s

$$0 \to K \xrightarrow{\text{inclusion}} H \xrightarrow{\beta} L \to 0$$

fin. gen

fin. gen $x_1, \dots, x_k$

fin. gen. $y_1, \dots, y_\ell$

pick $z_1 \in \beta^{-1}(y_1), \dots, z_\ell \in \beta^{-1}(y_\ell)$

Show: $x_1, \dots, x_k, z_1, \dots, z_\ell$ generate H

pick arbitrary $h \in H \rightsquigarrow \beta(h) = a_1 y_1 + \dots + a_\ell y_\ell$, where $a_i \in \mathbb{Z}$

$$\beta(h - a_1 z_1 - \dots - a_\ell z_\ell) = \beta(h) - a_1 \beta(z_1) - \dots - a_\ell \beta(z_\ell) = 0$$

$$\Downarrow$$

$$h - a_1 z_1 - \dots - a_\ell z_\ell \in \text{inclusion}(K)$$

$$\Downarrow$$

$$h - a_1 z_1 - \dots - a_\ell z_\ell = b_1 x_1 + \dots + b_k x_k \quad \text{, where } b_i \in \mathbb{Z}$$

$$\Downarrow$$

$$H \ni h = a_1 z_1 + \dots + a_\ell z_\ell + b_1 x_1 + \dots + b_k x_k$$

$\square$